Lesson 7: Oblique Triangles; Law of Sines; Law of Cosines; The Ambiguous Case; Area of a Triangle

Lesson Objectives:

- Use the Law of Cosines and the Law of Sines to find a missing side or angle of a given triangle
- Find the area of a triangle

Oblique Triangles

Previous to this we've only used the trigonometric functions to solve angles in right triangles, or things that could be translated into right triangles through the unit circles. In this lesson, we will learn how to use the trig functions to solve for any triangle.

First, we need to review a couple of facts about triangles. All triangles have angles that add up to 180°. In all triangles, the length of a side is related to the measure of the angle opposite it: the shortest side will always be across from the smallest angle, for example. However you can't tell much of anything about a triangle if you only know the angles...there are unlimited possibilities for the lengths of the sides! So to solve a triangle, you must know at least one side. Also, a side alone can't give you enough information to figure out all the ratios. This means that to solve a triangle, you will need at least three elements. Here are the possibilities, and what you will need to use to solve the triangle:

AAS: You are given two angles and a side opposite one of the given angles.

ASA: You are given two angles and the included side.

SSA: You are given two sides and the angle opposite one of them.

SAS: You are given two sides and the included angle.

SSS: You are given three sides.

AAA: It is not possible to solve a triangle given only the angles.

Now you can use a mixture of many different rules to solve these triangles, but it is generally thought that for the first three options (AAS, ASA, SSA) it is easiest to use the Law of Sines, and for the remaining possibilities (SAS, SSS) it is easiest to use the Law of Cosines. Others believe that while it is more complicated, it is best to use the Law of Cosines whenever possible, as it is less ambiguous than the Law of Sines. If a problem doesn't tell you to use one over the other, the choice is yours--in some cases it may be best to use both.

Always remember that when you have two angles, you know three angles, because the angles must add up to 180°.

Law of Sines

One of the ways to solve for the sides of an oblique triangle is to use the Law of Sines.

Here's what the formula looks like:

$\sin A/a = \sin B/b = \sin C/c$

$a/\sin A = b/\sin B = c/\sin C$

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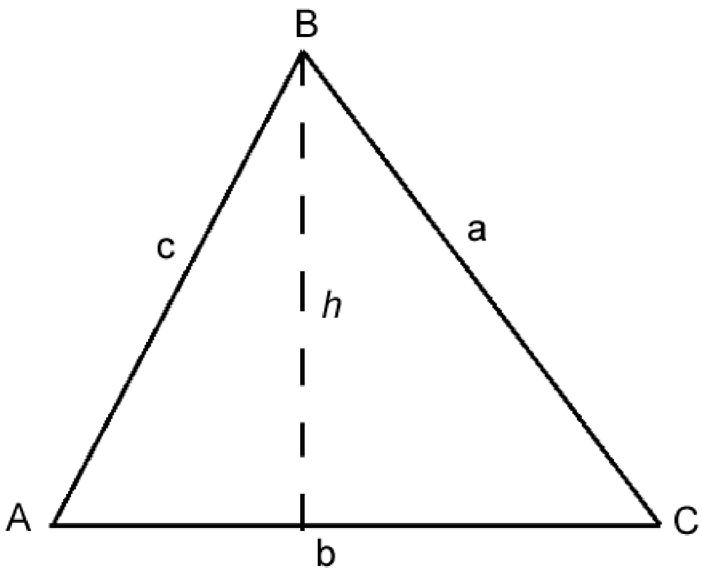

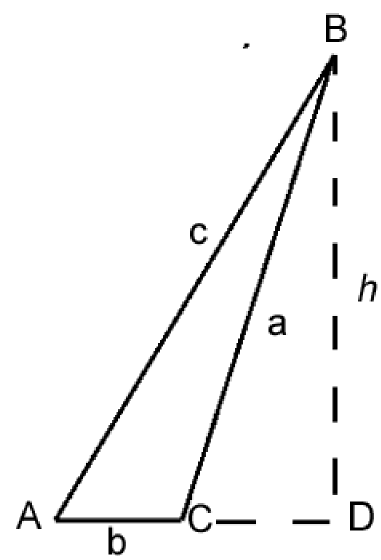

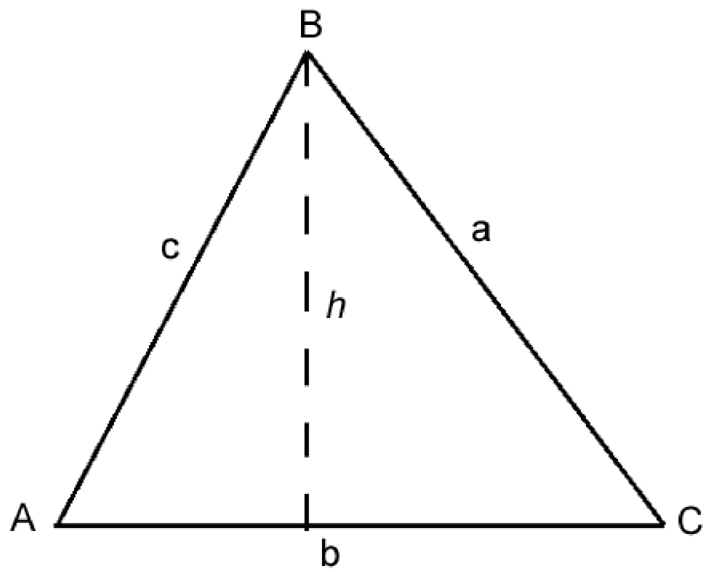

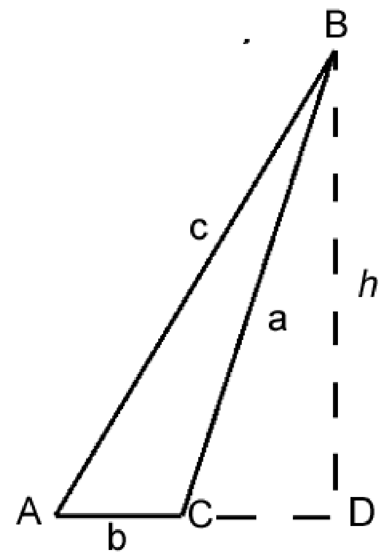

Now in both of these situations, you wind up with two right triangles. One created by side "c", a base, and the height, and one created by side "a", a base, and the height.

Using sine, you only need to know the opposite side and the hypotenuse. So in each of these you can create two things that must be equal to each other. In the first triangle, sinA=h/c and sinC = h/a...solving both of these statements for h, you get h=sinA(c) and h=sinC(a). Set them equal to one another and you get sinA(c) = sinC(a). Divide each side by "a" and "c", and you get sinA/a = sinC/c, which is part of the Law of Sines.

In the second triangle, the logic is slightly trickier: angle BCD has the same reference angle as angle C. We know this because sine is the same in the first and the second quadrant--so any two angles that add up to 180° will have the same sine value. The end result is the same however, we wind up with sinA/a = sinC/c. If you turn any of these triangles so a different side is on the bottom, and you can draw a different height, you can get the same result for sinB/b.

What this means in use is that if you have three items from any of the combinations, you can solve for the fourth, by setting up a ratio, like so:

sinA/a = sinB/b

sinA=(sinB/b)*a

a/sinA = b/sinB

a=(b/sinB)*sinA

sinA/a = sinC/c

sinA=(sinC/c)*a

a/sinA = c/sinC

a=(c/sinC)*sinA

sinB/b = sinA/a

sinB=(sinA/a)*b

b/sinB = a/sinA

b=(a/sinA)*sinB

sinB/b = sinC/c

sinB=(sinC/c)*b

b/sinB = c/sinC

b=(c/sinC)*sinB

sinC/c = sinA/a

sinC=(sinA/a)*c

c/sinC = a/sinA

c=(a/sinA)*sinC

sinC/c = sinB/b

sinC=(sinB/b)*c

$c/\sin C = b/\sin B$

$c=(b/\sin B)*\sin C$

The only difficulty is that because, as we mentioned above, sine is positive in both the first and second quadrants, you will occasionally run into a situation with two solutions. We will discuss this more fully in the section "The Ambiguous Case" below.

The Law of Cosines

The Law of Cosines flows more from the Pythagorean Theorem (actually, it is an extension of the distance formula, but we won't be proving it here). Some people consider it more difficult to use than the Law of Sines, because it requires more calculation. Here's what the formula looks like:

$a^2=b^2+c^2-2bc\cos A$

$b^2=a^2+c^2-2ac\cos B$

$c^2=a^2+b^2-2ab\cos C$

$\cos A=[b^2+c^2-a^2]/2bc$

If you are given three sides, you will need to use the Law of Cosines to solve for an angle. You can then use that information to use the Law of Sines to find the other angles.

The Ambiguous Case

There is a situation, generally when you know two sides and an non-inclusive angle (SSA), where you can find two answers for a triangle.

Say you are told that you have a triangle where side a=6, side b=8, and angle A=40°. You probably won't be offered a sketch, as that might give it away (or else the sketch will direct you toward one answer without realizing there are two). You are asked to solve the triangle, find the other sides and angles.

Using the Law of Sines you can set up the equation:

sinB=(sinA/a)*b

sinB=(sin40°/6)*8

sinB=(0.6428/6)*8

sinB=0.8571

$\sin^{-1}0.8571=58.99°$

But we've learned in previous lessons that the trigonometric functions are all good for more than one angle. Now the triangle needs its angles to add up to 180°, so our choices are not unlimited. But if 58.99° is a valid angle, so is 121.01°, an angle in quadrant II.

So far we have two possibilities, a triangle with angles 40°, 58.99°, and 81.01°, or a triangle with angles 40°, 121.01° and 18.99°.

How is this possible? Here's an image to help.

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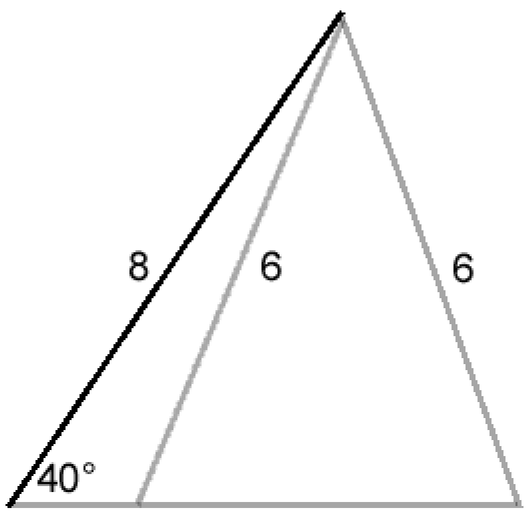

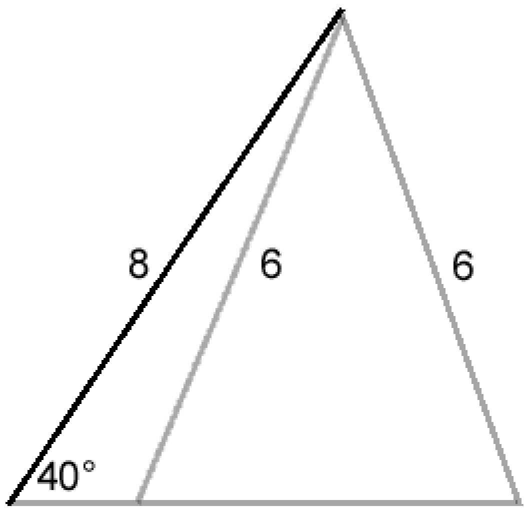

Keep in mind what was said before, about angles and the sides opposite them being related. We know that angle A is 40 degrees, and side a is 6. Since side b is 8, we know angle B must be bigger than 40°. Any solution we got where angle B was smaller than 40 degrees would be immediately dismissed. (This is one way to dismiss some of the ambiguities.) Also, sometimes one answer will be required, like you will be asked to find the shortest distance for c, or it will be required to create a support for a leaning beam (which would require a greater distance for c). This check will sometimes result in a triangle with no solution.

If you've been given enough clues to know which is the preferred answer, then you can solve for the side c that fits what you need. However if you don't know, then you'll need to solve for both possibilities.

Area of a Triangle

Of course, you are familiar with the traditional way to find the area of a triangle: $^1/_2$(b*h). Using the formulas above you have many new ways to find the sides you need in order to find the area of a triangle using the trigonometric functions. I have listed just a few of these here, but there are other possibilities: Take the time to study the relationships.

If two angles and a side are known:

$$\text{area of a triangle} = (a^2\sin B\sin C)/(2\sin A)$$

If two sides and the included angle are known:

$$\text{area of a triangle} = {}^1/_2 ab\sin C$$

But there is also another option, Heron's Formula, for when you know the three sides of a triangle (and you don't need to know anything more). Heron's Formula states:

$$\text{area of a triangle} = \sqrt{[s(s-a)(s-b)(s-c)]}$$

where s is half the perimeter (called the semiperimeter) of the triangle, and a, b, and c are the lengths of the three sides.

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Grading for this lesson:

- **10** is for a perfect set of answers the first time, or after one immediate revision. (see below)
- **9** is for 1, 2, or 3 mistakes on the original submission. In this case, you have an option: you can accept the 9, **or** you can post one revision within one week after the original submission was graded. If you correct all errors in this revision, your grade will be changed to a 10. In either case, you can proceed to the next lesson.

Students who miss more than 3 questions on their first posting must submit revisions until all problems are correct. Students should not submit incomplete revisions, or revisions without the work shown.

- **9: If you missed 4 or 5 questions on your original submission, you will receive a 9 after you have answered all questions correctly. You have unlimited revision attempts, but revisions must show your work.**
- **9: If you missed more than 5 questions on your original submission, you will receive a 9 if you have answered all questions correctly within 5 revisions, not including questions that you post to the teacher.**

Any questions should be substantive:
"I don't understand" doesn't help the instructor help the student; the student needs to ask specific questions. If a student cannot form a question to ask, they may instead post three examples with work shown to demonstrate what they are doing (and what they need help with) without it counting against the revisions.

- **8 will be given if you missed more than 5 questions on your original submission and you have answered all questions correctly. You have unlimited revision attempts, but revisions must show your work.**

Grades may be lowered by one point for when the student is repeatedly not paying attention to instruction, for lack of effort on the part of the student, or for rude or inappropriate responses.

- **7 will be given when the student is repeatedly not paying attention to instruction, for continuing lack of effort on the part of the student, or for repeated rude or inappropriate responses. All revisions must still be completed.**
- **6 reserved for administrative use**
- **5 will be given for cheating and plagiarism - purposeful or mistaken which will lower your final grade for the course (so be very careful when posting your work!) Revisions**

and reworking may still be required of the student, even though it will not improve the grade.

No lesson is complete without the approval of the instructor, and all revisions must be completed before a grade is assigned. No grade will be given for incomplete work.

Assignment

Complete the following questions. If you get more than three wrong, you will be asked to resubmit the wrong answers and show your new work. The teacher will give you advice on what you are doing wrong.

Solve the following, show your work:

1. In a triangle ABC, if A = 153°, C = 15°, and b = 11, find B.

2. In a triangle ABC, if A = 153°, C = 15°, and b = 11, find c.

3. In a triangle ABC, if a = 7, b = 8, and c = 2, find A.

4. In a triangle ABC, if a = 7, b = 8, and c = 2, find C.

5. In a triangle ABC, if a = 12, b = 7, and c = 14 find B.

6. In a triangle ABC, if a=12, b=17, and B=35°, find A.

7. In a triangle ABC, if b = 10, B = 30°, and C = 70°, find A.

8. In a triangle ABC, if b = 10, B = 30°, and C = 70°, find a.

9. In a triangle ABC, if a = 10, b = 12, and c = 14 solve for A.

10. In a triangle ABC, if a = 10, b = 12, and c = 14 solve for B.

11. What is the area of the triangle ABC if a = 4, b = 7, and B = 75°?

12. What is the area of the triangle ABC if a = 10, b = 12, and c = 14?

13. What is the area of the triangle ABC if a = 10, b = 14, and c = 20?

14. What is the area of the triangle ABC if a = 7, b = 8, and c = 2?

15. What is the area of the triangle ABC if a = 12, b = 7, and c = 14?

16. What is the area of the triangle ABC if a=12, b=17, and B=35°?

17. What is the area of the triangle ABC if A = 153°, C = 15°, and b = 11?

18. A surveyor intends to create a bridge across a river. There is a tall tree on the other side of the river. He measures a line down his side of the river for 125 feet. At each side of this line he uses his surveying equipment to measure the angle to the tree on the other side of the river. At the beginning of the line, the angle to the tree was 65°10', at the end of the line the angle to the tree is 60°10'. If he wants the bridge to go from where he started his line to the tree, then how long will the bridge be?

19. Two hikers are wandering through heavy woods with walkie talkies. The walkie talkies have a range of 100 yards. From their starting point, they head off at an angle of 109°10' of each other. Hiker 1 walks 0.24 miles per hour, hiker 2 walks 0.17 miles per hour. If each continues to go straight, how long will it be before they can no longer communicate?

20. A tree is growing at an angle out of the ground, 15°17' from vertical. Standing 10 feet away from the base, the angle to the top of the tree is 25°17'. What is the length of the tree?